Generalized Betchov relations for vector fields and their applications in turbulent flows

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For homogeneous incompressible turbulence, Betchov[1] proved that $\left\langle\overline{\mathbf{m}^2}\right\rangle = \left\langle\overline{\mathbf{m}^3}\right\rangle = 0$ for the velocity gradient $\mathbf{m} = \nabla\mathbf{u}$, in which $\overline{\mathbf{A}}$ denotes the trace of a second-rank tensor $\mathbf{A}$ and $\langle\rangle$ denotes the ensemble average. These two invariant relations provide important constraints on the statistics of $\mathbf{m}$, *e.g.* by relating strain self-amplification (SSA) and vortex stretching (VS). The result of Betchov is consequence of general relations, for gradients of arbitrary, homogeneous vectors: $\mathbf{h^a} = \nabla\mathbf{a}$, $\mathbf{h^b} = \nabla\mathbf{b}$ and $\mathbf{h^c} = \nabla\mathbf{c}$

$$\left\langle\overline{\mathbf{h^a h^b}}\right\rangle = \left\langle\overline{\mathbf{h^a}}\;\overline{\mathbf{h^b}}\right\rangle, \tag{1}$$

$$\left\langle\overline{\mathbf{h^a h^b h^c}}\right\rangle + \left\langle\overline{\mathbf{h^a h^c h^b}}\right\rangle = \left\langle\overline{\mathbf{h^a h^b}}\;\overline{\mathbf{h^c}}\right\rangle + \left\langle\overline{\mathbf{h^b h^c}}\;\overline{\mathbf{h^a}}\right\rangle + \left\langle\overline{\mathbf{h^c h^a}}\;\overline{\mathbf{h^b}}\right\rangle - \left\langle\overline{\mathbf{h^a}}\;\overline{\mathbf{h^b}}\;\overline{\mathbf{h^c}}\right\rangle. \tag{2}$$

We show here that Eqs. (1,2) provide new insight in several types of homogeneous turbulent flows:

Compressible turbulence.[2] In a compressible flow, taking $\mathbf{h^a} = \mathbf{h^b} = \mathbf{h^c} = \mathbf{m}$ yields $\left\langle\overline{\mathbf{m}^2}\right\rangle = \langle\overline{\mathbf{m}}^2\rangle$ and $\left\langle\overline{\mathbf{m}^3}\right\rangle = \frac{3}{2}\left\langle\overline{\mathbf{m}^2}\,\overline{\mathbf{m}}\right\rangle - \frac{1}{2}\langle\overline{\mathbf{m}}^3\rangle$. These relations could help determine all the second- and third-order moments of $\mathbf{m}$ experimentally in isotropic flows. Interestingly, numerical evidence show that they hold approximately even in a compressible mixing layer.

Pressure Hessian.[3] In incompressible turbulence, choosing $\mathbf{h^a} = \mathbf{h^b} = \mathbf{m}$ and $\mathbf{h^c} = \nabla\nabla p$, the pressure Hessian yields $\langle\overline{\mathbf{m}\mathbf{h^p}\mathbf{m}}\rangle = -\frac{1}{2}\left\langle\overline{\mathbf{m}^2}\right\rangle$, which imposes restrictions on the closure models for the pressure Hessian in the dynamics of $\mathbf{m}$. Together with the Poisson equation for the pressure, it also provides an integral expression for the fourth-order moment of velocity gradient in isotropic flows.

Filtered velocity gradient tensor.[4] **Choosing** $\mathbf{h^a} = \mathbf{h^b} = \nabla\tilde{\mathbf{u}}^{\ell_1}$ and $\mathbf{h^c} = \nabla\tilde{\mathbf{u}}^{\ell_2}$, with $\tilde{\mathbf{u}}^{\ell_1}$ and $\tilde{\mathbf{u}}^{\ell_2}$ being the filtered velocity at scales ℓ_1 and ℓ_2, we obtain a relation among invariants of filtered velocity gradients at two different filter sizes. Based on this relation we can show analytically that SSA contributes more to energy transfer than VS in the inertial range of incompressible turbulence, which supports the recent observation from numerical simulation of isotropic turbulence[5]. Moreover, our analytical proof requires only homogeneity, not restricted to isotropic turbulence.

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