Isotropy and a finite dimensional eigenvalue problem from the Lundgren hierarchy of turbulence

S. Görtz*†, J. Conrad* , D. Plümacher* and M. Oberlack*†

Our modern understanding of turbulence is focused on its statistical behavior. We therefore analyze Lundgren's [1] infinite but linear multi-point hierarchy of probability density functions (PDF). The problem of homogeneous isotropic tubulence (HIT) is revisited as an (quasi)exact solution of the LMN hierarchy. This hierarchy emerges directly from the Navier Stokes equations, with the PDF defined as follows: Any averaged function Q of the (statistical) physical velocities ${}_1\vec{u}, \ldots, {}_n\vec{u}$ at points ${}_1\vec{x}, \ldots, {}_n\vec{x}$ can be described via the integration of the n-point PDF ${}_nf$ over the sample space velocities ${}_1\vec{v}, \ldots, {}_n\vec{v}$

$$\langle Q({}_1\vec{u}, \ldots, {}_n\vec{u})\rangle = \int_{\mathbb{R}^{3n}} Q({}_1\vec{v}, \ldots, {}_n\vec{v})\, {}_nf\,(t, {}_1\vec{x}, {}_1\vec{v}, \ldots\, {}_n\vec{x}, {}_n\vec{v})\, \mathrm{d}_1\vec{v} \ldots \mathrm{d}_n\vec{v} \tag{1}$$

To account for the physical properties of HIT, we introduce spherical coordinates and obtain a dimensional reduction since each of the isotropic PDFs then only depends on time t, the spherical radius r, the radial sample space velocity component v^r and the absolute value of the sample space velocity components orthogonal to the radial comoponent, i.e. $\left(v^{\perp}\right)^2 = \left(v^{\theta}\right)^2 + \left(v^{\phi}\right)^2$. Due to linearity of the hierarchy, we formulate an ansatz of superposed products for the PDF, reading

$$_nf = \sum_{m=1}^{\infty} \prod_{k=1}^{n} {}_kh\left(t,\, {}_kr,\, {}_kv^r,\, {}_kv^{\perp};\, \lambda_m\right) \tag{2}$$

The superposition of eigenfunctions with eigenvalues λ_m is necessary since it takes into account the statistical coupling of processes on different scales. The Lundgren hierarchy comes with a number of side conditions [2]. These are consequently reduced for the case of isotropy. Applying the permutation condition on (2) greatly simplifies the hierarchy of equations, since it yields the same functional form for all ${}_kh$. With this simplification, the initially infinite dimensional hierarchy is reduced to a *single* but *non-linear integro-differential* equation, given by

$$\frac{\partial h}{\partial t} + v^r \frac{\partial h}{\partial r} = -\frac{\partial h}{\partial v^\nu} \int \nabla^\nu G(\vec{x}, \tilde{\vec{x}}) \tilde{r}^2 \sin(\tilde{\phi}) \int \left((\tilde{v}^r)^2 \frac{\partial}{\partial \tilde{r}^2} + \frac{1}{r} (\tilde{v}^\perp)^2 \frac{\partial}{\partial \tilde{r}} \right) \tilde{h}\ \mathrm{d}\tilde{\vec{v}}\, \mathrm{d}\tilde{\vec{x}}, \tag{3}$$

where Einstein's summation convention was used and the viscous term was neglected for reasons of shortness. The resulting equation together with the associated side conditions forms an eigenvalue problem for the eigenvalues λ_m, since the eigenfunctions have to be superposed in order to fulfill all side and boundary conditions, such as ${}_nf$ decaying to zero for ${}_k\vec{v} \to \pm\infty$.

We further give an outlook on approaches to obtain a solution. We state that the dimensionally reduced system admits additional symmetries, which allow further insight into new scaling laws and the underlying turbulence physics.

*Chair of Fluid Dynamics, TU Darmstadt, Germany

†Graduate School Computational Engineering, TU Darmstadt, Germany

[1] Lundgren, *Phys. Fluids*, **10**, 969 (1967)

[2] Hosokawa, *Phys. Rev. E*, **73**, 067301 (2006)