Symmetries in Second Moment Turbulence Modeling

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Attempts to model turbulence date back to the late 19th century, with notable works by Boussinesq[1] and Reynolds[2]. Many turbulence models have since emerged, including eddy viscosity, Reynolds stress, and combinations of both. However, their limitations often stem from missing symmetries, as seen in Prandtl's mixing-length model[3]. Efforts to generalize turbulence models by removing explicit spatial dependencies are thus at the same time efforts to include more symmetries into the turbulence model.

Symmetries are known to most as geometrical properties, e.g. rotation applied to a circle. In this context, symmetries are variable transformations that leave a given equation form invariant. The symmetries of the incompressible Navier-Stokes equation include time and pressure translation, rotation, generalized Galilean transformation, and scaling. The generalized Galilean symmetry allows for the transformation of independent variables under constant acceleration.

Simple turbulence models based on the Boussinesq assumption require a flow-dependent turbulent length scale, which usually compromises Galilean symmetry. Two-equation models address this issue by introducing a second scale-providing variable, but still have more symmetries than the exact Navier-Stokes equation, being invariant under constant rotations[4].

In the late 1940s and early 1950s, turbulence models known as Reynolds stress models (RSMs) were developed. These models avoid the Boussinesq approximation by modeling the unclosed terms of the transport equation for Reynolds stresses. Most models have the same symmetries as the exact Navier-Stokes equations and are hence more general than two-equation models.

Recently, new symmetries of the infinite set of multi-point correlation equations and other exact statistical descriptions of turbulence such as the Lundgren probability density function hierarchy have been found[5]. These are dubbed statistical symmetries, as they only occur in the statistical descriptions of turbulence and have been linked with intermittency and non-Gaussianity - critical properties of turbulence[6].

All turbulence models covered so far are not invariant under these statistical symmetries. Only recently, a barebone RSM that is invariant has been proposed[7].

We show very concretely how a RSM can be developed from all the symmetries known so far, and further we also give a recently developed model. With the symmetries thus available, we show that many classical but, in particular, also the latest near-wall turbulent scaling laws[8] can be reproduced with the new model, which exhibit clearly intermittent behavior.

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