A linear stochastic model of turbulent cascades and fractional fields

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In three-dimensional Navier-Stokes turbulence, energy is injected at a large scale L and is efficiently transported to small scales. In this process, the fluid reaches a state of finite variance and large spatial gradients, which can be approximately described by a rough velocity field of Hölder exponent $H \approx 1/3$. Motivated by this phenomenon, this work describes a stochastic partial differential equation (SPDE) for a d-dimensional scalar field u that is randomly stirred by a spatially smooth and uncorrelated in time forcing term. Previous studies [1,2] described how fractional complex fields in one spatial dimension can be obtained through the action of a linear transport operator. This construction is formalized and extended to arbitrary dimensions with the following transport equation in Fourier space,

$$\partial_t \widehat{u}(t,k) + \mathrm{div}_k \left(\frac{ck\widehat{u}(t,k)}{|k|} \right) + c\frac{H+\frac{1}{2}}{|k|}\widehat{u}(t,k) = \widehat{f}(t,k). \tag{1}$$

In this equation, the term with a divergence is responsible for a cascading transfer of energy from large to small scales [3] [4], and the term with H for the development of fractional regularity of corresponding order. Linear transport of energy in fluids is an effect first observed in the focusing of waves onto attractors which takes place in rotating and stratified flows [5] [6] and later described as the general effect of homogeneous operators of 0 order [3]. A complete characterization of the solution is given: It is smooth at any finite time, and converges to a fractional Gaussian field at infinite time, up to smaller-order corrections. High-resolution spectral simulations of this SPDE with added viscous dissipation are also performed, and their solutions reach a stationary state in finite time which reproduces the asymptotic analytical results for the power spectral density and increment statistics, with the development of a self-similar inertial range between the forcing and dissipative scales.

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