On the conditional stability of shear flows

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It is well-known that the unconditional stability limit of flows can be calculated using the Reynolds-Orr equation. However, the predicted limits (Re_E) are usually too conservative compared to experimental results[1]. The trivial alternative method is linear stability analysis, which gives a Reynolds number limit (Re_L) for an unconditionally unstable flow. Unfortunately, this limit is often high and impractical.

A possible solution to this problem is the usage of generalized kinetic energy. We can describe the perturbed flow by a nonlinear ordinary differential equation obtained by Galerkin projection. The vector q_i represents the coefficients of the basis, and the kinetic energy of the perturbation is $e = q_i q_i$. The nonlinear terms of the differential equation do not influence the time derivative of e, according to the Reynolds-Orr identity, and the unconditional stability limit can be obtained.

However, a more generalized definition of kinetic energy can be given as $h = r_i r_i$, where r_i is the transformed state vector fulfilling $q_i = S_{i,j} r_j$ and $S_{i,j}$ is a transformation matrix. The time derivative of h depends on the amplitude, and the stability $\mathrm{d}h/\mathrm{d}t < 0$ can be proved until a certain level of $h_{\mathrm{crit}}(Re)$. The lowest amplitude perturbation state, where the $\mathrm{d}h/\mathrm{d}t$ becomes 0, is called the critical perturbation, and its energy level is e_{crit}. However, e_{crit} is not the stability limit, since the transformation $S_{i,j}$ can be non-orthogonal. We must calculate the state with the smallest kinetic energy (e_{min}) whose generalized kinetic energy is equal to the critical one (h_{crit}). It can be proved that if $e < e_{\mathrm{min}}$, the flow is stable.

The method is applied to Walaffe's dynamical model[2] using the original parameters ($\lambda = \mu = \sigma = 10, \nu = 15$). The $S_{i,j}$ matrices were optimized at each Reynolds number to maximize e_{min}. The values of e_{min} and e_{crit} as the function of Reynolds number are shown in Fig. 1. Both functions decrease as expected. This method can be applied to real flow configurations to obtain their conditional stability limit.

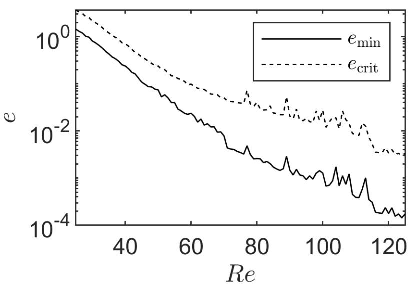

Figure 1: The energy condition limit (e_{min}) and the energy of the critical perturbation (e_{crit}) as the function of Reynolds number.

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[1] Nagy, *Phys. Rev. E* **105 (3)**, 035108 (2022).

[2] Waleffe, *Phys. Fluids* **7**, 3060 (1995).